Math 144 Lecture Notes 11-10-09

1. Discussion Problems

Problem 1. Show, using the well-ordering of $\mathbb{N}$ and definition by induction, that if X is an infinite set, there is a $1-1$ map from X into itself which is not onto.

Proof. Take any element of X and pair it with 1 and call it x_1, take a second element of X and pair it with 2 and call it x_2. Continue this process and you will, since X is infinite, associate to each element of $\mathbb{N}$ an element of X. By doing this we are creating a bijection from a subset $A = \{x_i | i \in \mathbb{N}\}$ with $\mathbb{N}$ in its well-ordering, thus A is well-ordered. Define a map $f : A \to A$ by:

$$f(x_n) = x_{2n}$$

then clearly f is $1-1$ and is not onto as x_n where n is odd is not in the image of f. Now extend f to a $1-1$ function $\tilde{f}$ on X by:

$$\tilde{f}(x) = \begin{cases} f(x) & x \in A \\ x & x \notin A \end{cases}$$

Then $\tilde{f}$ is $1-1$ but not onto.

$\square$

Problem 2. Prove that if X is a set, then there cannot be a map from X onto $\wp(X)$.

Proof. Assume $\exists\ f : X \to \wp(X)$ which is onto. Thus to any element $a \in A$ there is an associated $f(a)$ ($f(a) \subset A$). Now we have two possibilities: $a \in f(a)$ or $a \notin f(a)$. Let B be the subset of A consisting of all a such that $a \notin f(a)$. Like any subset of A, $B = f(a)$ for some $a \in A$. Suppose $\exists\ b \in A$ such that $f(b) = B$. Then the question is: Does b lie in B? If $b \in B$, by the definition of B, $b \notin f(b)$, which is not possible. Thus $b \notin B$. But again this is a problem because, by the definition of B, $b \in f(b)$, which is again a contradiction. Thus there can exist no map from X onto $\wp(X)$. $\square$

2. Cardinal Arithmetic [1]

2.1. **Countability.** A *countable set* is a set that has a bijection with the natural numbers. An *uncountable set* is an infinite set that is not countable.

2.2. **Cardinal Numbers.** Write $A \sim B$ if there is a bijection between A and B. This is clearly an equivalence relation on sets (Exercise: prove it!). Thus there is a partition generated by this equivalence relation. Assign to each equivalence class a number called a *cardinal number*. By this definition, two sets have the same cardinal number if there is a bijection between them (i.e. $card(A) = card(B)$). We can think of a cardinal number, n, as being the number of elements in the set which has cardinal number n. Thus you might be tempted to say $card(\mathbb{N}) = \infty$. This is wrong, however because in this same way you would end up with $card(\mathbb{R}) = \infty$, and, as we have shown, there is no bijection between $\mathbb{N}$ and $\mathbb{R}$. We define $card(\mathbb{N}) := \aleph_0$.

For now, call $card(\mathbb{R}) = \mathfrak{c}$. We can define higher cardinal numbers, but will do that later.

Definition 1. *Let d and e be cardinal numbers. Let D and E be sets with $card(D) = d$ and $card(E) = e$. We say that $d \leq e$ if there exists a one-to-one function from D into E. We say $d < e$ if $d \leq e$ and $d \neq e$.*

Theorem 1. *(Cardinality of the Power Set) Let D be a set with $card(D) = d$. Then $card(\wp(D)) = 2^d$.*

Theorem 2. *(Schröder-Bernstein) Let A and B be sets such that there exists a one-to-one map of A into B and a one-to-one map of B into A. Then there exists a bijection between A and B.*

2.3. **Cardinal Addition.** Let d and e be cardinal numbers. To define $d + e$ we take disjoint sets D and E with cardinalities d and e respectively. Then we define $d + e := card(D \cup E)$.

Theorem 3. *Let d and e be the cardinal numbers of the sets D and E respectively. Suppose $d \leq e$, $d \neq 0$, and e infinite. Then $d + e = e$.*

2.4. **Cardinal Multiplication.** Let d and e be cardinal numbers. Let D and E be sets with cardinalities d and e respectively. Then define $de := card(D \times E)$.

Theorem 4. *Let d and e be the cardinal numbers of the sets D and E respectively. Suppose $d \leq e$, $d \neq 0$, and e infinite. Then $de = e$.*

2.5. **Cardinal Exponentiation.** Let d and e be nonzero cardinal numbers. Let D and E be sets with cardinalities d and e respectively. Then define $d^e := card(D^E)$ where D^E is the set of all function from E into D.

Theorem 5. *(Cardinality of $\mathbb{R}$) Using this definition of cardinal exponentiation, and the fact that there is a bijection between $\mathbb{R}$ and $2^{\mathbb{N}}$ (i.e. $card(\mathbb{R}) = card(2^{\mathbb{N}})$) we can define the cardinality of $\mathbb{R}$ by $card(\mathbb{R}) = card(2^{\mathbb{N}}) = 2^{\aleph_0} = \mathfrak{c}$, where we view 2 as $\{0, 1\}$.*

Property 1. *(Cardinal Exponentiation) Let d_1, d_2, d, e_1, e_2, e, and f be cardinal numbers. Then:*

(1) $(d_1 d_2)^e = d_1^e d_2^e$
(2) $d^{e_1+e_2} = d^{e_1} d^{e_2}$
(3) $(d^e)^f = d^{ef}$

3. Infinite Cardinal Numbers [1]

We can list the well-ordered set of infinite cardinal numbers as follows

$$\aleph_0, \aleph_1, \aleph_2, ..., \aleph_n, ...$$

Where does $2^{\aleph_0} = \mathfrak{c}$ fit into this list? We can write the complete well-ordered set of cardinal numbers:

$$0, 1, 2, 3, ..., n, ..., \aleph_0, \aleph_1, \aleph_2, \aleph_3, ..., \aleph_n, ..., \aleph_\omega, \aleph_{\omega+1}, ...$$

Definition 2. *A limit cardinal is a cardinal number that has no immediate predecesor.*

The first limit cardinal is $\aleph_0$, the second is $\aleph_\omega$, the third is $\aleph_{2\omega}$, and so on...

4. The Continuum Hypothesis [1]

Consider the well ordered sequence of infinite cardinals

$$\aleph_0, \aleph_1, \aleph_2, ...$$

Where does $\mathfrak{c} = 2^{\aleph_0}$ fit into this sequence? Since $\aleph_0$ corresponds to a countable set, we know that $\mathfrak{c} \geq \aleph_1$.

Conjecture 1. *(Continuum Hypothesis)* $2^{\aleph_0} = \aleph_1$

Conjecture 2. *(Generalized Continuum Hypothesis)* $2^{\aleph_n} = \aleph_{n+1}$

Remark 1. *An interesting thing to note is that it was proved by Gödel in 1938 that the Continuum Hypothesis is unable to be disproved, then in 1963 Cohen proved that it cannot be proved. So, as it stands, the Continuum Hypothesis is undecidable on the basis of the current axioms for set theory.*

In practice we assume the Continuum Hypothesis because it makes cardinal arithmetic much easier and it seems consistant with what we expect.

References

[1] Kaplansky, I., *Set Theory and Metric Spaces*, Chelsea Publishing Company, New York, (1977).